1. (6 points) Find the equation of the tangent line to the graph of $y = \dfrac{6}{x^3} + 8x$ at $x = 1$.

$f(x) = 6x^{-3} + 8x \implies f'(x) = -\dfrac{18}{x^4} + 8$ +2

$f(1) = 6 + 8 = 14$ +1 $\quad f'(1) = -18 + 8 = -10$ +1

$\implies y = f(1) + f'(1)(x-1) = 14 - 10(x-1)$

+1 +1

2. (4 points) Find the exact value (i.e. without using a calculator) of $p'(2)$ where $p(x) = \dfrac{\ln(x)}{x^3}$.

$p'(x) = \dfrac{\frac{1}{x} \cdot x^3 - \ln(x) \cdot (3x^2)}{x^6} \implies p'(2) = \dfrac{4 - 12\ln(2)}{64}$

+1 quotient

+1 dtop

+1 dbot

+1 answer

3. A bookstore is examining the cost (in thousands of dollars) of storing q cubic meters of books and determined it follows the function

$$C(q) = q^4 e^{-2q} + q$$

(a) (4 points) Find the marginal cost function.

$C'(q) = 4q^3 e^{-2q} + q^4(-2e^{-2q}) + 1$

+1 $\frac{d}{dq}(q)$

+1 product

+1 each derivative in product rule

(b) (4 points) Find the marginal cost if 3 cubic meters of books are stored. Include units and round your answer to three decimal places.

$C'(3) = 4(27)e^{-6} + 81(-2e^{-6}) + 1 = -\frac{5}{e^6} + 1 = 0.866$ thousands of \$ per m^3

+2 plug in 3

+1 units

+1 answer

or 866 \$/cubic meter

4. (3 points) The demand function for the latest iPhone is expressed as $q = D(p)$, where q is measured in thousands of units sold, and p is the price in hundreds of dollars. Explain in practical terms the meaning of the statement $D'(6) = -223.8$. Include appropriate units.

When the iphone is priced at 600 \$, the number of units sold is decreasing at a rate of 223.8 thousand units per hundred dollars, i.e. that if we increased the price by 100 \$ we expect to sell 223,800 fewer units

+1 "when priced"

+1 rate decreasing

+1 units

5. A local Girl Scout troop has been looking at the sales numbers of their cookies and extrapolated the following table of data, representing the amount $q = D(p)$ of boxes sold as a function of the given price point p (measured in dollars).

p	5.5	5.75	6	6.25	6.5	6.75	7
q	2765	2440	1980	1660	1175	800	430

(a) (4 points) Find the average rate of change of the demand function $q = D(p)$ on the interval $[5.5, 7]$. Explain the real-life meaning of your answer in an English sentence, including appropriate units.

the rate is $\frac{D(7)-D(5.5)}{7-5.5} = \frac{430-2765}{1.5} = -1556.\bar{6}$ units/\$

+1 +1 +1

+1 When increasing the price from 5.5 to 7 dollars per box, they will on average sell 1557 fewer boxes per dollar of price increase.

(b) (6 points) Estimate the elasticity of demand at the price $p = 6$. Explain the real-life meaning of your answer in an English sentence, including appropriate units.

+1

$$E = \left|\frac{p}{q}\cdot\frac{dq}{dp}\right| \approx \left|\frac{p}{q}\cdot\frac{\Delta q}{\Delta p}\right| = \left|\frac{6}{1980}\cdot\frac{1660-1980}{6.25-6}\right| = |-3.87| = 3.87$$

+1 estimate $\frac{dq}{dp}$ w/ RoC

+1 +1 +1

+1

When cookies are sold at 6 \$/box, increasing the price by 1% will decrease the amount of boxes sold by 3.87%

Note: this is still part of question 5, and uses the same demand function $q = D(p)$ given from the table.

(c) (2 points) Is the demand at $p = 6$ elastic or inelastic? If we increased the price, would our revenue increase or decrease? Explain in a sentence.

Since $E \approx 3.87$, demand is elastic +1

Since demand is elastic, increasing price would decrease revenue. +1

6. A parent of a newborn child wishes to open a college fund. Suppose that the account is opened today and has a continuous interest rate of 7.1%. Determine the constant amount S that must be invested each year so that the account will contain \$216,000 in 18 years.

$$FV = \int_0^M S(t)\, e^{r(M-t)}\, dt = \int_0^{18} S \cdot e^{0.071(18-t)}\, dt = 216{,}000$$

+2 +1

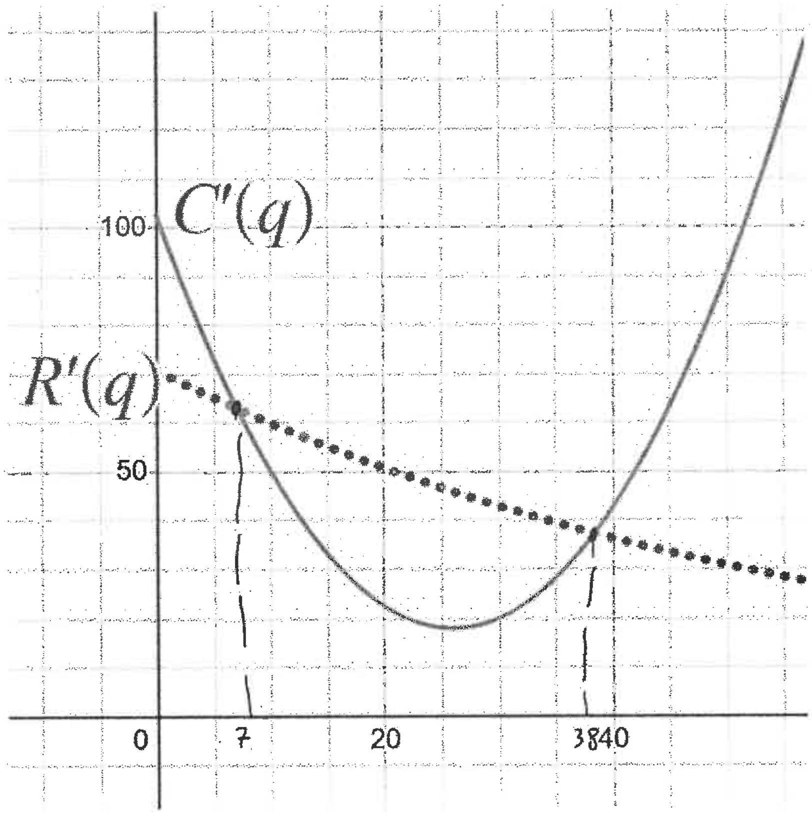
$$\Rightarrow S \cdot \int_0^{18} e^{0.071(18-t)}\, dt = 216{,}000$$

+1

$$\Rightarrow S = \frac{216{,}000}{\int_0^{18} e^{0.071(18-t)}\, dt} = 5922.48\ \$$$

+1 +1

7. (4 points) The graph below represents the **MARGINAL** cost $C'(q)$ and **MARGINAL** revenue $R'(q)$ of selling q units of an item.



Use the graph to determine the q-value of maximum profit, explaining your answer in a sentence below.

$\pi'(q) = R'(q) - C'(q) \Rightarrow$

+1 estimate values where $\pi' = 0$

7 38

$-$ + $-$ $\pi'(q)$

min max

+1

+1

At $q \approx 38$ $\pi'(q)$ changes from + to $-$, showing $\pi(q)$ has a local max there

8. Consider the function
$$f(x) = x^6 - 9x^4.$$

(a) (4 points) Find all critical points of $f(x)$.

+1 +1

$f'(x) = 6x^5 - 9(4x^3) = 6x^5 - 36x^3 = 6x^3(x-6)$

$\Rightarrow f'(x) = 0$ if $x = 0$ or $x = 6$

+1

180 × 22
90 44
45 88
22 196
11 392

(b) (4 points) For each of the critical points found in part (a), determine if it is a local maximum, local minimum or neither. You may use either the first or second derivative test to justify your answer.

5 784
2 1568
3136

FDT

0 6

+ ↑ − ↑ + $f'(x)$

max min

$x = 0$ is max
1 $x = 6$ is min

SDT $f''(x) = 30x^4 - 96x^2 \Rightarrow f''(0) = 0$ inconclusive

$f''(6) = 35424 > 0$

$\Downarrow$

$x = 6$ is min

+1 setup
+1 testing points
+1 answer consistent w/ sign line
+1 signs correct

Parts (c) and (d) below are a continuation of problem 7, and they refer to the same function

$$f(x) = x^6 - 9x^4$$

used in parts (a) and (b).

(c) (4 points) Find the absolute (or global) maximum and absolute (or global) minimum of $f(x)$ on the interval $-1 \le x \le 4$.

+1 +1

Critical #s are $x = 0, 6 \Rightarrow$ Test $x = -1, 0, 4$ (6 is out of domain)

$f(-1) = 1 - 9 = -8 \leftarrow$ global min

$f(0) = 0$

$f(4) = 1792 \leftarrow$ global max

+1 plug into $f(x)$

+1 answer

(d) (4 points) Find all value(s) of x for which $f''(x) = 0$ and determine whether or not these value(s) are inflection points.

+1

$f''(x) = 30x^4 - 96x^2 = x^2(30x^2 - 96)$

+1

$\Rightarrow x = 0$ and $x = \pm\sqrt{\frac{96}{30}} = \pm 1.79$

-1.79 0 1.79

$+$ $-$ $-$ $+$ $f''(x)$

+1

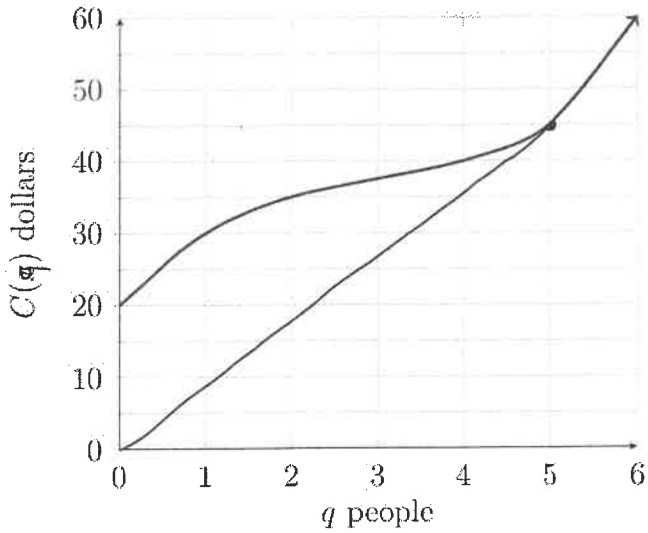
$\Rightarrow x = -1.79, 1.79$ are IP

$x = 0$ is not an ip

(it's a max

+1

9. (4 points) The graph below represents the total cost $C(q)$ in dollars of providing a service to q people.



Use the graph to estimate the q-value of minimum **average** cost $a(q)$, explaining your answer in a sentence below.

+2 $q \approx 5$ is the value of minimum average cost, since it's where the line through the origin is tangent to $C(q)$

+2

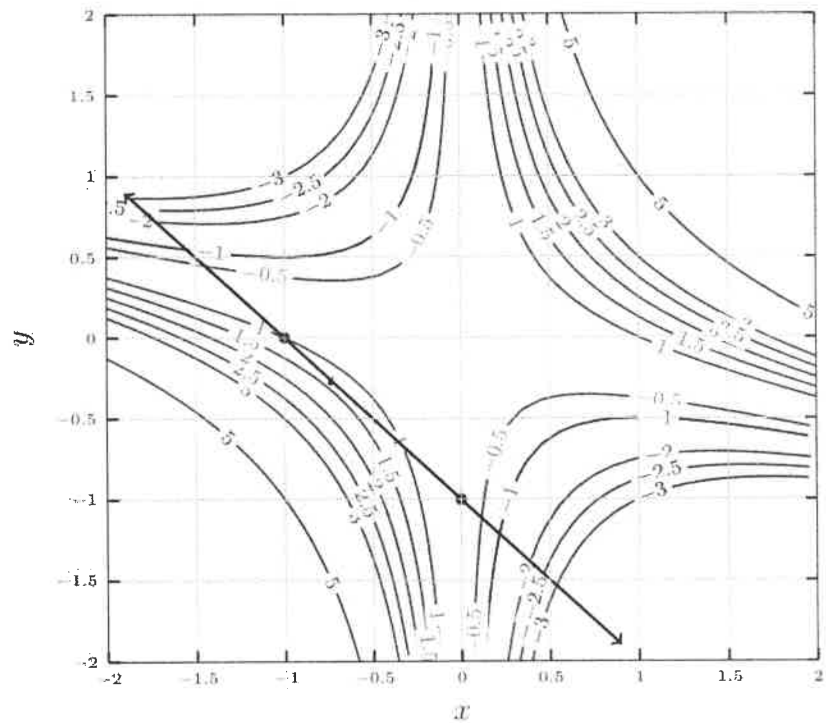
10. (4 points) Suppose that $f(x)$ is a function such that $\int_3^{11} f(x)\,dx = 24$. If $F(x)$ is an antiderivative of $f(x)$ and $F(3) = -15$, find the value of $F(11)$.

By the FTC, $F(11) = F(3) + \int_3^{11} f(x)\,dx = -15 + 24 = 9$

+2 +1 +1

11. A contour plot of $f(x, y) = x^2 + 4xy$ and the graph of the line $x + y = -1$ are given below. We wish to

$$\begin{array}{rl} \text{optimize} & f(x,y) = x^2 + 4xy \\ \text{subject to} & x + y = -1 \end{array}$$



(a) (4 points) Use the graph to estimate the maximum and minimum values of $f(x, y)$ subject to the constraint. From the bubble below, choose all that apply, filling the blanks in as necessary. You do not have to do any calculation.

- ○ There is no maximum value subject to the constraint. +1
- +1 ● Maximum of approximately 1.4 at $(x, y) \approx$ (-0.75, -0.25)
- +1 ● There is no minimum value subject to the constraint.
- ○ Minimum of approximately ________ at $(x, y) \approx$ ________

(b) (4 points) Write out the system of equations that you would solve to find the (x, y) points giving maximimum or minimum values of f subject to this constraint. You do not have to solve the system.

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = c \end{cases} \Rightarrow \begin{cases} 2x + 4y = \lambda(1) \\ 4x = \lambda(1) \\ x + y = -1 \end{cases}$$

+2 +2

12. Let $h(x,t) = 3e^{xt}$, and compute the following partial derivatives.

(a) (4 points) $h_t(x,t)$

$h_t(x,t) = 3xe^{xt}$

+2 x is constant

+2 power rule

(b) (4 points) $h_{tt}(x,t)$

$h_{tt}(x,t) = 3x\,(x e^{xt}) = 3x^2 e^{xt}$

+2 x is constant

+2 power rule.

(c) (4 points) $h_{tx}(x,t)$

$h_{tx}(x,t) = 3e^{xt} + 3x \cdot (te^{xt})$

+2 product

+1 derivative

each

13. Let $f(x, y) = x^3 + y^3 - 6x^2 - 12y + 10$.

(a) (4 points) Find all first- and second-order partial derivatives of f.

$f_x(x,y) = 3x^2 - 12x$ +1 $\quad f_y(x,y) = 3y^2 - 12$ +1

$f_{xx}(x,y) = 6x - 12$ +0.5 $\quad f_{yy}(x,y) = 6y$ +0.5

$f_{xy}(x,y) = 0 = f_{yx}(x,y)$

+1

(b) (3 points) Use your partial derivatives to find the critical points of f. Show your work.

$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 12x = 0 \\ 3y^2 - 12 = 0 \end{cases} \Rightarrow \begin{cases} 3x(x-4) = 0 \\ y^2 = \frac{12}{3} = 4 \end{cases}$

+1

$\Rightarrow \begin{cases} x = 0 \text{ or } x = 4 \\ y = \pm 2 \end{cases}$ +1 $\Rightarrow (0,2), (0,-2), (4,2), (4,-2)$

+1

(c) (3 points) Use your partial derivatives to identify each critical point you found as a local minimum, a local maximum, or neither using the second derivative test. Show your work.

$D(x,y) = (6x-12)(6y) - 0^2 = 36xy - 72y$

$D(0,2) = -144 < 0 \to$ saddle

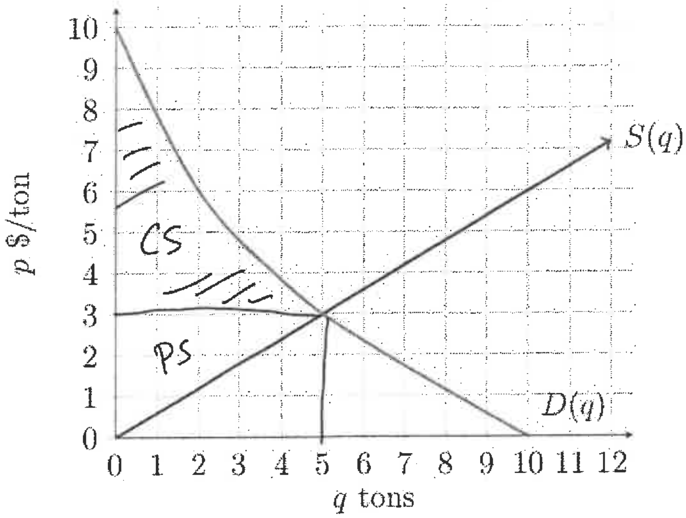
$D(0,-2) = 144, \quad f_{xx}(0,-2) = -12 \to$ max

$D(4,2) = 144, \quad f_{xx}(4,2) = 12 \to$ min

$D(4,-2) = -144 \to$ saddle

+1 method
+1 plugs points from (b)
+1 answer

14. The graph below shows supply and demand graphs for a good.



(a) (4 points) Use a Riemann sum with $n = 5$ subintervals to estimate $\int_0^5 D(q)\,dq$. Specify whether you're using left or right sums.

q	0	1	2	3	4	5
P	10	8	6	5	4	3

+2

+1 left sum : 10 + 8 + 6 + 5 + 4 = 33 +1

right sum : 8 + 6 + 5 + 4 + 3 = 26

(b) (2 points) Use geometry to find an exact value for $\int_0^5 S(q)\,dq$.

Area = $\frac{5 \cdot 3}{2}$ = 7.5

+1 +1

+1 only if swapped
+1 each

(c) (2 points) What is the equilibrium price and quantity for this market? $(p^*, q^*) =$ (3, 5)

(d) (1 point) Shade the region on the graph which corresponds to the consumers surplus. +1

(e) (3 points) Use the at least one of the values you've computed in parts (a) and (b) to estimate the value of the consumer surplus.

$CS = \int_0^5 D(q)\,dq - p^* q^*$ =

+1 +1
33 − 7.5 = 25.5 (left)

26 − 7.5 = 18.5 (right)

+1

Formula Page

Derivatives:

- $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$
- $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$
- $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$
- $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
- $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
- $\frac{d}{dx}(c) = 0$, if c is a constant
- $\frac{d}{dx}(mx+b) = m$, where m, b are constants
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
- $\frac{d}{dx}\left(a^x\right) = a^x \cdot \ln(a)$
- $\frac{d}{dx}\left(e^x\right) = e^x$
- $\frac{d}{dx}\left(e^{kx}\right) = k \cdot e^{kx}$, if k is a constant
- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

Other things:

- Quadratic formula:
  $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Future and present value, yearly compounding: $FV = PV \cdot (1+r)^t$
- Future and present value, continously compounding: $FV = PV \cdot e^{rt}$
- Elasticity: $E = \left|\frac{p}{q} \cdot \frac{dq}{dp}\right|$
- Consumer surplus: $\int_0^{q^*} D(q)\, dq - p^*q^*$
- Producer surplus: $p^*q^* - \int_0^{q^*} S(q)\, dq$
- Present value of income stream $S(t)$:
  $PV = \int_0^M S(t)e^{-rt}\, dt$
- Future value of income stream $S(t)$:
  $FV = e^{rM} \cdot PV = \int_0^M S(t)e^{r(M-t)}\, dt$
- Discriminant:
  $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$